Sedimentation and creaming are two analogous phenomena, but in one case, "particles" in the liquid have a density higher than the liquid, and in the other case, the density is lower. In other words, in one case, particles fall, but in the other case, they float. "Particles"? It can be any objects, from the subatomic particle to the planet, or even the galaxy... but frequently, in the context of food, they are microscopic structures, in colloidal systems (do you know exactly what it is?).

The question

When a particle, or when many particles, are in liquid, they can sediment or cream, as said before. One would like to explore their motion, either the individual motion for one particle, or the collective motion for many particles.

Let's make a model, in one macroscopic case first

Let's consider a model: one particle put in a liquid. First, the particle is assumed to be much bigger than the molecules of the liquid, so that the liquid seems to be "continuous" (an assumption which we have to keep in mind for later)

Let's make a picture:

A good practice is to "simplify" the problem, dropping all secondary aspects of it (and this is why it is important to be able to distinguish different orders of magnitude). For example, here, the number of dimensions was reduced from 3 to 2. One can go further, assuming a spherical particle:
Please note that in this picture the particle was put in the centre of the liquid, on a vertical axis of symetry. This is useless, except for aesthetics.

Then, the problem has to be translated into calculation, introducing litteral variables : to this end, it is sufficient to say what is seen, i.e. a particule having a shape and a matter (shown on the picture by the color), and a liquid, having a matter.

Hence, for the shape of the particle, one variable is enough:
- radius \( r \) of the particle.

For the material of it:
- density of the particle \( \rho_p \)

For the liquid:
You can see that, for the liquid, the viscosity was introduced, because even children know that if there are motions in a liquid, the liquid has to move away in front of the liquid, whereas it has to stick back behind. The particle is drawing with it some liquid, and this is characterized by viscosity.

Finally, as there is a motion, and because we assume this motion in one direction, in a particular sense, we have to introduce a vertical axis, toward the bottom, for example (Oz).

Now we are ready for the search of relationships between the introduced variables.

**restart:**
Remember that we have to describe a sedimentation or a creaming, i.e. a motion. Hence, we have to consider the laws of dynamics.

The first law, the "fundamental law of motion", or "Newton's law", indicates: "The sum of forces acting on a body is equal to the product of the mass of this body by its acceleration".

This leads us very logically to observe that we did not introduce the mass \( m \) of the particle, and its velocity \( v \).

Which forces do act on the particle?

1. First the weight:
   \[ P := m g \]
   \[ (2.1) \]

2. Then Archimedes force, equal to the weight of the volume of liquid displaced by the body:
   \[ F := V \cdot \rho_l \cdot g \]
   \[ V \rho_l g \]
   \[ (2.2) \]

3. Finally, when the particle moves, there are frictions, which can be described using the "Stokes force":
   \[ S := 6 \cdot \pi \cdot \eta \cdot v(t) \cdot r \]
   \[ 6 \pi \eta v(t) r \]
   \[ (2.3) \]

Now, some new variables appear, such as the velocity \( v \). One can try to link them to variables already introduced.

For example, for the velocity:

\[ v(t) := \frac{d}{dt} z(t) \] 
\[ t \rightarrow \frac{d}{dt} z(t) \]

\[ (2.4) \]

We are now ready to write the fundamental law of motion, assuming a sedimentation:
Let's observe now that there is a relationship between the masse and the radius of the particle:

\[ m := \rho_p \cdot V \]

(2.6)

With :

\[ V := \frac{4}{3} \pi r^3 \]

(2.7)

Finally, here is the equation of the motion:

\[ eq1 ; \]

\[ \frac{4}{3} \rho_p \pi r^3 g - \frac{4}{3} \pi r^3 \rho_l g - 6 \pi \eta \left( \frac{d}{dt} \, z(t) \right) r = \frac{4}{3} \rho_p \pi r^3 \left( \frac{d^2}{dt^2} \, z(t) \right) \]

(2.8)

Let's take a break, in order to have time of being happy of this small success. Now, we have to solve this "differential equation". This is a differential equation of the second order, with only one unknown \( z \). Using a software such as Maple, it is very simple to get the solution of the equation... as we just need to ask !

\[ dsolve(eq1) \]

\[ z(t) = \frac{9}{2} \frac{\eta t}{\rho_p^2} \left( -C1 \right) - \frac{2}{9} \frac{r^2 g \left( -\rho_p + \rho_l \right) t}{\eta} \]

(2.9)

And here it is!

Let's observe the result. We see 3 terms... and 2 constants. How to find them ? In the problem, we did not consider the initial conditions. This is easy.

For example:

- we have to write that the initial velocity is zero; to this goal, we calculate the speed:

\[ \frac{d}{dt} \left( -\frac{9}{2} \frac{\eta t}{\rho_p^2} \left( -C1 \right) - \frac{2}{9} \frac{r^2 g \left( -\rho_p + \rho_l \right) t}{\eta} \right) \]

(2.10)
And we solve this equations:

\[
solve \left( e^{- \frac{9}{2} \frac{\eta t}{r^2 \rho_p}} \ - C1 - \frac{2}{9} \frac{r^2 g \left( -\rho_p + \rho_l \right)}{\eta} = 0, \ - C1 \right) \]

\[
- C1 := \frac{2}{9} \frac{r^2 g \left( -\rho_p + \rho_l \right)}{\eta} e^{- \frac{9}{2} \frac{\eta t}{r^2 \rho_p}} \eta \]

\[
\frac{2}{9} \frac{r^2 g \left( -\rho_p + \rho_l \right)}{\eta} e^{- \frac{9}{2} \frac{\eta t}{r^2 \rho_p}} \eta \]

Then, we use another condition, and we make the same process:

\[
solve \left( - \frac{2}{9} \frac{r^2 \rho_p e^{- \frac{9}{2} \frac{\eta t}{r^2 \rho_p}}}{\eta} \ - C1 - \frac{2}{9} \frac{r^2 g \left( -\rho_p + \rho_l \right)}{\eta} \ + C2 = 0, \ - C2 \right) \]

\[
\frac{2}{81} \frac{r^2 g \left( -\rho_p + \rho_l \right) \left( 2 r^2 \rho_p + 9 \eta t \right)}{\eta^2} \]

And here is the result...

But using Maple, you can make it much easier, much more elegant!

\[\text{Another way of getting the solution... without taking much care of the initial conditions}\]

For this second method, we begin as before, writing the equations which we have to solve:

\[
P := m g; \quad m g\]

\[
F := V \cdot \rho_l \cdot g; \quad V \rho_l g\]

\[
S := 6 \cdot \pi \cdot \eta \cdot v(t) \cdot r, \quad (2.1.2)\]
And as before, we get the equation of the motion:

\[eq1 := P - F - S = m \cdot \frac{d}{dt} v(t)\]

\[m g - V \rho_l g - 6 \pi \eta \left( \frac{d}{dt} z(t) \right) r = m \left( \frac{d^2}{dt^2} z(t) \right)\]

\[m := \rho_p \cdot V;\]

\[\rho_p V\]

\[V := \frac{4}{3} \pi r^3;\]

\[\frac{4}{3} \pi r^3\]

And as before, we get the equation of the motion:

\[eq1;\]

\[\frac{4}{3} \rho_p \pi r^3 g - \frac{4}{3} \pi r^3 \rho_l g - 6 \pi \eta \left( \frac{d}{dt} z(t) \right) r = \frac{4}{3} \rho_p \pi r^3 \left( \frac{d^2}{dt^2} z(t) \right)\]

However, here, the trick is to group the initial conditions in one variable called \textit{ics} (for example):

\[ics := z(0) = 0, D(z)(0) = 0;\]

\[z(0) = 0, D(z)(0) = 0\]

And we ask Maple to solve the differential equation using directly the initial conditions:

\[dsolve(\{eq1, ics\})\]

\[z(t) = \frac{4}{81} \frac{r^4 \rho_p e}{\eta ^2} g \left( -\rho_p + \rho_l \right) - \frac{2}{9} \frac{r^2 g \left( -\rho_p + \rho_l \right) t}{\eta} \]

\[+ \frac{4}{81} \frac{r^4 g \left( -\rho_p + \rho_l \right) \rho_p}{\eta ^2}\]

You see ? We get immediately the result.

Is it the end of the story? No! A validation is needed, in science!

\[\nabla\text{The result is explored}\]
I don't discuss the validation here, because I want to focus on how using a result, showing that one should not stop to a dry equation...

Let's start from the result:

\[
z(t) = - \frac{9}{2} \eta \frac{r^2 \rho_p}{r^2 \rho} \frac{\eta t}{C1} - \frac{2}{9} \frac{r^2 g (-\rho_p + \rho_l) t}{\eta} + C2
\]

You see 3 terms:
- a constant (the last term), quite uninteresting, because it tells only at which level the particle was dropped at time \( t = 0 \);
- a term of the shape \( \exp(-t) \) : it decreases very fast, so that it is important only at the onset of motion;
- a term proportional to \( t \), lasting, increasing linearly, very important at long times; after the start, it becomes the main one.

If the motion is almost regular, it means that the velocity is almost at constant speed, and the acceleration is progressively tending toward 0. The long times regime is called "stationary".

It can be reached when the acceleration is 0:

\[
eq 2 := \frac{4}{3} \rho_p \pi r^3 g - \frac{4}{3} \pi r^3 \rho_l g - 6 \pi \eta \left( \frac{d}{dt} z(t) \right) r = 0
\]

\[
\frac{4}{3} \rho_p \pi r^3 g - \frac{4}{3} \pi r^3 \rho_l g - 6 \pi \eta \left( \frac{d}{dt} z(t) \right) r = 0
\]

\[\text{dsolve(eq2)}\]

\[
z(t) = - \frac{2}{9} \frac{r^2 g (-\rho_p + \rho_l) t}{\eta} + C1
\]

Here, we get only the last two terms, and the limit velocity is:

\[
v_{\text{lim}} := \frac{d}{dt} \left( - \frac{2}{9} \frac{r^2 g (-\rho_p + \rho_l) t}{\eta} + C1 \right)
\]

\[
- \frac{2}{9} \frac{r^2 g (-\rho_p + \rho_l)}{\eta}
\]

One can see that:
- the more the difference in densities, the fastest the sedimentation (and this is why centrifugation is used, in labs and in the industry);
- the greatest the viscosity, the slowest sedimentation;
- the smallest radiuses are associated with slow sedimentation... but look that the effect is at the square!

You also observe that if you know 3 of the 4 parameters \((r, \rho_p, \rho_l, \eta)\) and if you measure the limit velocity, then you can determine the fourth!
Questions around the limit velocity

When does the particle reach this limit velocity? \textit{Stricto sensu}, never, but one can ask when it is 99 \% of it, or 99.9 \%, for example.

For more generality, let's use a velocity which is a fraction of the limit velocity:

\[ V := p \cdot v_{\text{lim}} \]

We now ask when this velocity is obtained:

\[
solve \left\{ \frac{d}{dt} \left( -\frac{2}{9} \frac{r^2 g \left( -\rho_p + \rho_l \right)}{\eta} \right) = -\frac{4}{81} \frac{r^4 \rho_p e^{-\frac{9}{2} \frac{\eta t}{r^2 \rho_p g \left( -\rho_p + \rho_l \right)}}}{\eta^2} \right. \\
- \frac{2}{9} \frac{r^2 g \left( -\rho_p + \rho_l \right) t}{\eta} + \frac{4}{81} \frac{r^4 g \left( -\rho_p + \rho_l \right) \rho_p}{\eta^2} \left. \right) t \right. \\
- \frac{2}{9} \frac{\ln\left( -p + 1 \right) r^2 \rho_p}{\eta} \right) \]  

(3.1.2)

Then, we explore this solution for spherical particles with a radius of 1 mm, of density four times the density of water, with viscosity \( \eta = 10^{-3} \text{ kg/m/s} \).

\[
\text{subs} \left( r = 0.001, \ \rho_l = 1000, \ \rho_p = 4000, \ p = 0.99, \ \eta = 0.001, \ -\frac{2}{9} \frac{\ln\left( -p + 1 \right) r^2 \rho_p}{\eta} \right) \\
-0.8888888888 \ln(0.01) \right) \]  

(3.1.3)

\text{evalf} (%) \]  

4.093484609 \right) \]  

(3.1.4)

We find the time: 4. Four what? Four seconds, as we always SI units.

With a lower density of particles:

\[
\text{evalf} \left( \text{subs} \left( r = 0.001, \ \rho_l = 1000, \ \rho_p = 1200, \ p = 0.99, \ \eta = 0.001, \ -\frac{2}{9} \frac{\ln\left( -p + 1 \right) r^2 \rho_p}{\eta} \right) \right) \\
1.228045383 \right) \]  

(3.1.5)

We see that the limit velocity is reached before.

A smaller radius for the particles?
Here, we see that the limit velocity is reached very fast... but there is something more: having done 3 calculations, it's beginning to be boring... and we also see that this exploration of the result is not systematic. Do you know how to make it better?

Let's dip in the microscopic world, now

It what we did before, we considered particles much bigger than water molecules (for example, with particles of radius 1 mm, the ratio is \( \frac{10e-3}{10e-10} = 1.000000000 \times 10^7 \), with is very big. But remember our assumption! Water is not a continuous medium, and when particles are smaller, then the water molecules would appear in the picture. And remember that water molecules move.

If water molecules bumping a stone don't make it move much, on the contrary they have a big effect for small and light particles. Did you remember the Brownian motion?

How can we describe phenomena? This asks for results of statistical thermodynamics.
Let's now imagine a vessel with two flat levels, or one step. One level will be called "up", and the other "down. Small particles (red) are bumped by water molecules, so that some of the particles of the up level can go down, and some of the particles of the down level can go up. Of course, as there is gravity, one could imagine that there are more particles on the bottom than on up level (for particles having a density much than water), but how many?
Thermodynamics is considering equilibria:
At the equilibrium (I shall not give the demonstration here), the "Boltzmann distribution" tells you the proportion of particles of mass \( m \) at the level \( h \) (for example, \( h \) being counted from the bottom). Very broadly, this distribution compares the potential energy \( E \) and the thermal motion energy \( k_B \cdot T \), where \( k_B \) is the Boltzmann constant, and \( T \) the temperature (in kelvins, K). For example, the Boltzmann distribution can be used in order to determine the number of spins in a level up, when the spins are in a magnetic field. For example, this distribution... can apply to thousands of particular cases.. when you know it.

But let's come back to our question of sedimentation and creaming. Thanks to the Boltzmann distribution, we can calculate the number of particles at the height \( h \):

\[
n(h) = K \exp \left( -\frac{m \cdot g \cdot h}{k \cdot T} \right)
\]

Here \( K \) is a constant. How much is \( K \)? We need an equation to calculate it, and this equation is the mathematical expression of the fact that the sum of particles at all levels is equal to the total number of particles in the system \( N \).

In other words:

\[
N = \int_0^H n(h) \, dh
\]

Symbolic calculus software such as Maple are wonderful, because they make it faster and easier to explore mathematical problems. Let's choose all variables equal to 1, in order to display the law:

\[
\text{plot}\left(1 \cdot \exp \left( -\frac{1 \cdot 1 \cdot h}{1 \cdot 1} \right), \, h = 0 \ldots 100\right)
\]

No big surprise to observe a decreasing exponential!

Using now realistic values, in SI units:

\[
g \:= 9.8 : \\
k \:= 1.38 \times 10^{-23} : \\
T \:= 300 : \\
\rho \:= 1 \times 10^3 : \\
m \:= \frac{\rho \cdot 4 \cdot \pi \cdot r^3}{3} :
\]
And, again, we could go on increasing masses, decreasing them, etc. in order to make a bridge with the former question, for example.

\[ n := h \to \frac{1}{K} \cdot \exp\left(-\frac{m \cdot g \cdot h}{k \cdot T}\right); \]
\[ r := 1e-6; \]
\[ K := \int_{0}^{+\infty} \exp\left(-\frac{m \cdot g \cdot h}{k \cdot T}\right) \, dh; \]
\[ \text{plot}(n(h), h = 0..1e-6) \]

\[ h \to e^{-\frac{m \cdot g \cdot h}{k \cdot T}} \]
At the XXI th century

The first differential equations were at the lever of an engineer of the XIX th century. Then, Boltzmann calculation were at the level of engineers of the XX th century. But remember that you live in the XXI rst century. How should you move now? If you want to "innovate" in your company, you need to know more than others before, and you need to apply modern knowledge. What was introduced since Boltzmann? Many things, but in particular quantum physics and numerical methods, such as dynamic combinatorial, for example! How would you apply them to this sedimentation and creaming question?